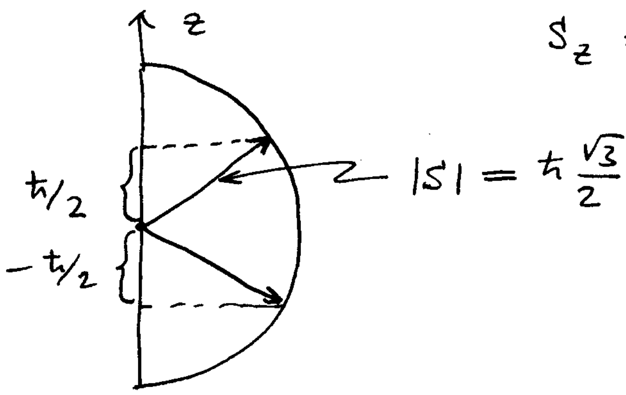
Spin: The electron has an intrinsic angular momentum with the quantum number $s = 1/2$.

Magnitude of spin:

$$|\vec{S}| = \sqrt{S^2} = \hbar\sqrt{s(s+1)} = \hbar\frac{\sqrt{3}}{2}$$

Projections of spin: $m = s, s-1, \dots, -s = 1/2, -1/2$

$$S_z = \pm\frac{\hbar}{2}$$



Matrix representation:

For the angular momentum operator:

$$\langle l, m' | \hat{L}_z | l, m \rangle = \hbar m \,\delta_{m,m'}$$

$$\langle 1/2, m' | \hat{S}_z | 1/2, m \rangle = \hbar m \,\delta_{m,m'}$$

$m' = 1/2, \quad m = 1/2 \rightarrow \frac{\hbar}{2}$

$m' = 1/2, \quad m = -1/2 \rightarrow 0$

$$\longrightarrow \quad S_z = \hbar \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

Raising/Lowering operators:

$$\langle l, m' | L_+ | l, m \rangle = \hbar \sqrt{l(l+1) - m(m+1)}\ \delta_{m', m+1}$$

$$\langle \tfrac{1}{2}\ m' | S_+ | \tfrac{1}{2}, m \rangle = \hbar \sqrt{\tfrac{3}{4} - m(m+1)}\ \delta_{m', m+1}$$

$m' = \frac{1}{2}, \quad m = \frac{1}{2} \quad \rightarrow \quad 0$

$m' = \frac{1}{2}, \quad m = -\frac{1}{2} \quad \rightarrow \quad \hbar \sqrt{\frac{3}{4} + \frac{1}{4}} = \hbar$

$m' = -\frac{1}{2}, \quad m = \frac{1}{2} \quad \rightarrow \quad 0$

$$S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Similarly,

$$S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Relation to S_x, S_y: $\quad S_+ = S_x + i S_y$

$\quad S_- = S_x - i S_y$

$$S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y = \frac{1}{2i}(S_+ - S_-) = \frac{\hbar}{2i}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

<u>Pauli matrices</u>:

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Commutation relations:

$$[\sigma_x, \sigma_y] = 2i\sigma_z , \qquad \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \overleftrightarrow{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_x\sigma_y + \sigma_y\sigma_x = 0$$
$$\sigma_z\sigma_x + \sigma_x\sigma_z = 0$$
$$\sigma_y\sigma_z + \sigma_z\sigma_y = 0$$

$\leftarrow$ anticommutation property, applies only to $s = 1/2$

Spin eigenstates:

Spinor is a two-column eigenvector of S_z

$$S_z \begin{pmatrix} u \\ v \end{pmatrix} = \pm \frac{\hbar}{2} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \pm \begin{pmatrix} u \\ v \end{pmatrix}$$

"+" : $u = u$
$v = -v \rightarrow v = 0$

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

"−" : $u = -u$
$v = v$

$$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

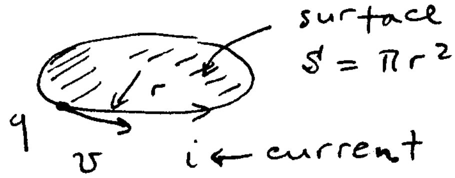
An arbitrary state of spin, spinor, can be expanded in χ_+ and χ_-:

$$\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = \alpha_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_- \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Normalization: $|\alpha_+|^2 + |\alpha_-|^2 = 1$

$|\alpha_+|^2, |\alpha_-|^2 \leftarrow$ probabilities that a measurement of S_z will give either $\hbar/2$ or $-\hbar/2$

10-2 Intrinsic magnetic moment



$$M = i S, \quad i = \frac{qv}{2\pi r}$$

$$M = \frac{qv}{2\pi r} \times \pi r^2 = \frac{qvr}{2} = \frac{qL}{2m}$$

Generally: $\vec{M} = \frac{q\vec{L}}{2m} = -\frac{e\vec{L}}{2m}$ ← This is true for orbital motion

↑ electron is negative

For spin, one needs the gyromagnetic ratio g:

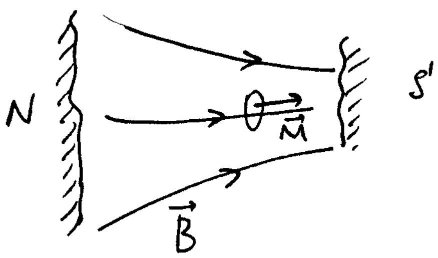
$$\vec{M} = -g\frac{e\vec{L}}{2m}, \qquad g \approx 2$$

↗ electron mass

For proton: $\vec{M} = g\frac{e}{2m_p}\vec{L}$

nuclear g-factor ↗ ← proton mass

Energy in the magnetic field:



$$E = -\vec{M}\cdot\vec{B}$$

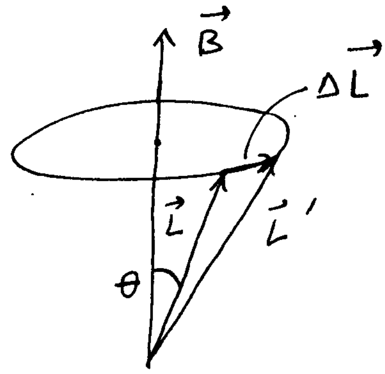
magnetic dipole is attracted to stronger magnetic field, paramagnetic effect

Precession of magnetic moments:

Atomic magnet placed in the magnetic field will precess

Torque acting on a dipole

$$\vec{\tau} = \vec{M} \times \vec{B}$$



$$\frac{d\vec{L}}{dt} = \vec{\tau} = M \times \vec{B}$$

$$\frac{dL}{dt} = \omega_p L \sin\theta = MB \sin\theta$$

$$\omega_p = g \underbrace{\frac{eB}{2m}}_{\omega_c}$$

ω_c cyclotron frequency

"it is <u>not</u> possible to understand the magnetic effects of materials in any honest way from the point of view of classical physics", R. Feynman

<u>What happens with quantum magnet?</u>

Hamiltonian:

$$H = -\vec{M} \cdot \vec{B} = \frac{eg\hbar}{4m_e} \vec{\sigma} \cdot \vec{B}$$

Spinor state:

$$\psi(t) = \begin{pmatrix} \alpha_+(t) \\ \alpha_-(t) \end{pmatrix} = e^{-i\omega t} \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}$$

$$\hbar\omega \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = \frac{eg\hbar B}{4m_e} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}$$

B parallel to z

$$\omega_0 = \frac{egB}{4m_e}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\omega_0 = -\frac{egB}{4m_e}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

two possible solutions

Classical analogy:

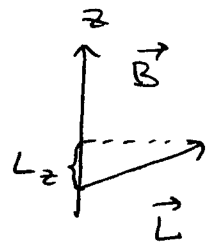
Larmor frequency: $\omega_L = \frac{eB}{2m}$ (compare to $\omega_0 = \frac{egB}{4m}$)

Larmor theorem:

The motion of charges in a weak magnetic field is the same as without it with an added rotation, about the axis of the field, with the angular velocity ω_L.

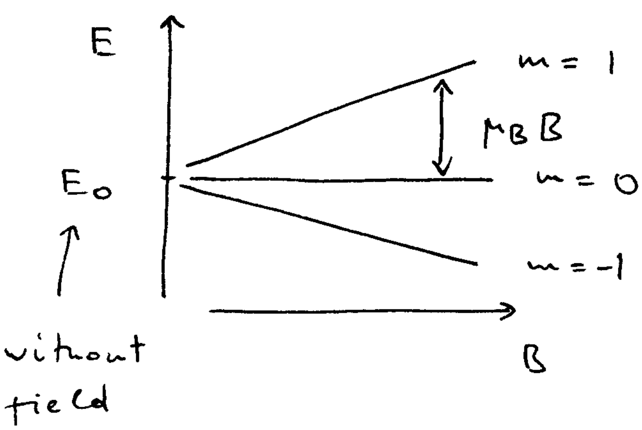
Energy of a magnet

$\vec{M} = -g\frac{e}{2m}\vec{L}$, $\quad \mathbf{H} = g\frac{e}{2m} L_z B_z$

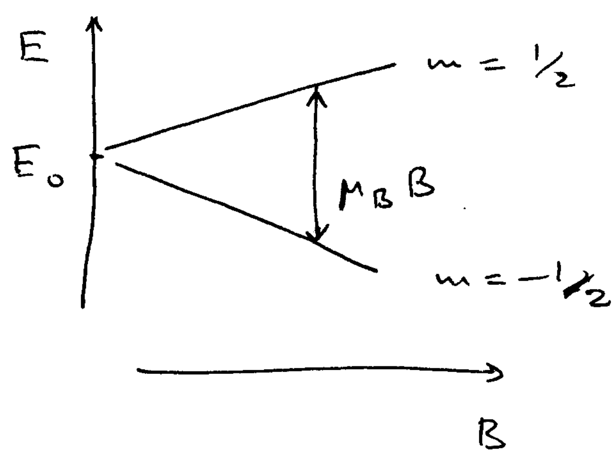


$$\langle l, m | H | l, m\rangle = \frac{g e \hbar}{2m} B_z\, m$$

In case of $l = 1$:

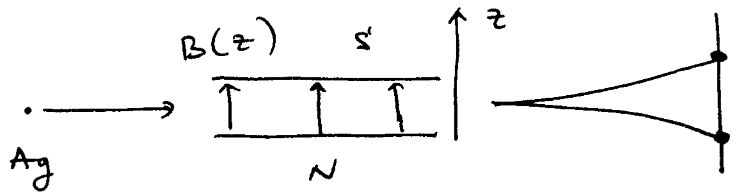


For electron:



$\mu_B = \frac{e\hbar}{2m}$ ← Bohr magneton

Back to Stern - Gerlach experiment



quantum result: only two projections of the spin

classical picture

$$F_z = -\frac{\partial U}{\partial z} = \mu \cos\theta \frac{\partial B}{\partial z}$$

all possible orientations relative to the field

10-4 Addition of angular momenta

Total angular momentum of two particles

$$\vec{J} = \vec{L}_1 + \vec{L}_2$$

Total angular momentum satisfies the same rule as individual angular momenta: one can define only one projection and the total value:

$$J^2 \psi_{Jm_j} = \hbar^2 j(j+1)\psi_{Jm_j}, \quad J_z \psi_{J,m_j} = \hbar m_j \psi_{jm_j}$$

two quantum numbers

$$\psi_{jm_j} = \alpha Y_{\ell_1 m_1} + \beta Y_{\ell_2 m_2}$$

linear combination of spherical harmonics of individual particles

$$j = \ell_1 + \ell_2, \; \ell_1 + \ell_2 - 1, \; \dots, \; |\ell_1 - \ell_2|$$

$$m_j = j, j-1, \dots, -j$$

Addition of two spins

$$\vec{S} = \vec{S}_1 + \vec{S}_2$$

$$[S_x, S_y] = [S_{1x} + S_{2x}, S_{1y} + S_{2y}] =$$

$$= [S_{1x}, S_{1y}] + \underbrace{[S_{2x}, S_{1y}]}_{=0} + \underbrace{[S_{2x}, S_{1y}]}_{=0} + [S_{2x}, S_{2y}] =$$

$$= i\hbar (S_{1z} + S_{2z}) = i\hbar S_z$$

$$[S_x, S_y] = i\hbar S_z \quad \leftarrow$$ same commutation rules as for the components

Spinors of the two-electron system:

$$\left.\begin{aligned} S_1^2 \chi_\pm^{(1)} &= \hbar^2 \tfrac{1}{2}\left(1+\tfrac{1}{2}\right)\chi_\pm^{(1)} \\ S_{1z}\, \chi_\pm^{(1)} &= \hbar\left(\pm \tfrac{1}{2}\right)\chi_\pm^{(1)} \end{aligned}\right\} \leftarrow \text{electron 1}$$

similar relations for $\chi_\pm^{(2)}$ of electron 2

Two-electron system has for spinors:

$$\chi_+^{(1)}\chi_+^{(2)}, \qquad \chi_+^{(1)}\chi_-^{(2)}, \qquad \chi_-^{(1)}\chi_+^{(2)}, \qquad \chi_-^{(1)}\chi_-^{(2)}$$

$$\uparrow\uparrow \qquad\qquad \uparrow\downarrow \qquad\qquad \downarrow\uparrow \qquad\qquad \downarrow\downarrow$$

z-projection:

$$S_z\, \chi_+^{(1)}\chi_+^{(2)} = (S_{1z} + S_{2z})\, \chi_+^{(1)}\chi_+^{(2)} =$$

$$= \chi_+^{(2)} S_{1z} \chi_+^{(1)} + \chi_+^{(1)} S_{2z} \chi_+^{(2)} = \hbar\, \chi_+^{(1)}\chi_+^{(2)}$$

Similarly,

$\uparrow\downarrow \quad S_z \chi_+^{(1)} \chi_-^{(2)} = 0 \qquad \downarrow\downarrow \; S_z \chi_-^{(1)} \chi_-^{(2)} = -\hbar \chi_-^{(1)} \chi_-^{(2)}$

$\downarrow\uparrow \quad S_z \chi_-^{(1)} \chi_+^{(2)} = 0$

The state with $m = \pm 1$ $(\uparrow\uparrow, \downarrow\downarrow)$ is called <u>triplet</u> and corresponds to parallel electrons.

Why do we get two states with $m = 0$?

Let's apply the lowering operator to the state of two spins up:

$$S_- = S_{1-} + S_{2-} = (S_{1x} + S_{2x}) - i\,(S_{1y} + S_{2y})$$

$$S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$S_- \chi_+ = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar \chi_-$$

$$S_- \chi_+^{(1)} \chi_+^{(2)} = \left(S_{1-} \chi_+^{(1)}\right) \chi_+^{(2)} + \chi_+^{(1)} S_{2-} \chi_+^{(2)} =$$

$$= \hbar \sqrt{2} \; \frac{\chi_+^{(1)} \chi_-^{(2)} + \chi_-^{(1)} \chi_+^{(2)}}{\sqrt{2}} = \hbar \sqrt{2}\, X_+$$

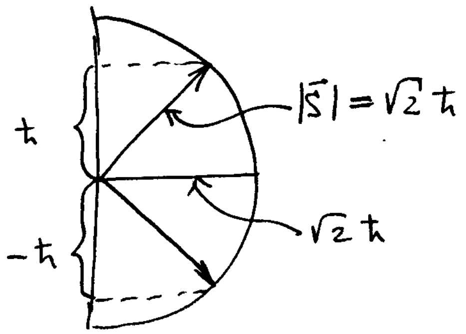
Let's now apply S_- to the new state X_+:

$$S_- X_+ = \hbar \sqrt{2}\, \chi_-^{(1)} \chi_-^{(2)}$$

S_- <u>does not</u> change the magnitude of S, only the projection, that is state $\chi_+^{(1)} \chi_+^{(2)}$, X_+, $\chi_-^{(1)} \chi_-^{(2)}$ belong to the same S

$\chi_+^{(1)}\chi_+^{(2)}$, X_+, $\chi_-^{(1)}\chi_-^{(2)}$ make three projections of the triplet state, $S = 1$.

$$|\vec{S}| = \hbar\sqrt{1(1+1)} = \sqrt{2}\,\hbar$$



Let's test it:

$$S_z X_+ = \frac{1}{\sqrt{2}}(S_{1z} + S_{2z})(\chi_+^{(1)}\chi_-^{(2)} + \chi_-^{(1)}\chi_+^{(2)}) =$$

$$= \frac{1}{\sqrt{2}}(\chi_+^{(1)}\chi_-^{(2)} - \chi_+^{(1)}\chi_-^{(2)} + \chi_+^{(2)}\chi_-^{(1)} - \chi_+^{(2)}\chi_-^{(1)})$$

$$= 0$$

Projection of X_+ on z-axis is indeed zero!

What is the <u>singlet</u> state ($S = 0$)?

$$X_- = \frac{1}{\sqrt{2}}(\chi_+^{(1)}\chi_-^{(2)} - \chi_-^{(1)}\chi_+^{(2)})$$

$$S_z X_- = \frac{1}{\sqrt{2}}(\chi_+^{(1)}\chi_-^{(2)} - \chi_+^{(1)}\chi_-^{(2)} + \chi_-^{(1)}\chi_+^{(2)} - \chi_-^{(1)}\chi_+^{(2)})$$

$$= 0$$

$$S^2 = S_1^2 + S_2^2 + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+}$$

$$S_1^2 X_- = \frac{\hbar^2}{\sqrt{2}}\left(\frac{3}{4}\chi_+^{(1)}\chi_-^{(2)} - \frac{3}{4}\chi_-^{(1)}\chi_+^{(2)}\right) = \frac{3}{4}\hbar^2 X_-$$

$$S_2^2 X_- = \frac{3}{4}\hbar^2 X_-$$

$$S_{1z}S_{2z}X_- = \frac{1}{\sqrt{2}}\left(\underbrace{S_{1z}\chi_+^{(1)}}_{+\hbar/2}\underbrace{S_{2z}\chi_-^{(2)}}_{-\hbar/2} - \underbrace{S_{1z}\chi_-^{(1)}}_{-\hbar/2}\underbrace{S_{2z}\chi_+^{(2)}}_{\hbar/2}\right) =$$

$$= -\frac{\hbar^2}{4}X_-$$

$$(S_{1+} S_{2-} + S_{1-} S_{2+}) \chi_- = -\hbar^2 \chi_-$$

$$S^2 \chi_- = \hbar^2 \left(\frac{3}{4} + \frac{3}{4} - \frac{2}{4} - 1 \right) \chi_- = 0 = \hbar^2 s(s+1) \chi_-$$

Therefore, $S = 0$ and χ_- corresponds to the singlet state.

<u>Summary</u>: Two electrons are characterized by <u>four</u> spinors:

z

triplet		singlet	
$\uparrow\uparrow$	$\chi_+^{(1)} \chi_+^{(2)}$	$\uparrow\downarrow$	$\frac{1}{\sqrt{2}} \left(\chi_+^{(1)} \chi_-^{(2)} - \chi_-^{(1)} \chi_+^{(2)} \right)$
$\rightrightarrows$	$\frac{1}{\sqrt{2}} \left(\chi_+^{(1)} \chi_-^{(2)} + \chi_-^{(1)} \chi_+^{(2)} \right)$		
$\downarrow\downarrow$	$\chi_-^{(1)} \chi_-^{(2)}$		