Uncertainty Relations

In our discussion of wave packets in Chapter 2, we noted that there is a relationship between the spread of a function and its Fourier transform. When the de Broglie correspondence between wave number and momentum is made, the relationship takes the form

$$\Delta p \Delta x \geq \hbar$$

What we called the spread or, in the above context, the uncertainty can be sharpened mathematically to a definition: The uncertainty in any physical variable $\Delta A$ is equal to the dispersion, given by

$$\Delta A^2 = \langle (A - \langle A \rangle)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2$$  \hspace{1cm} (5A-1)

With this definition we can show that the uncertainty relation follows directly from quantum mechanics. Let us consider two hermitian operators $A$ and $B$, and shift them by a constant, their expectation values in an arbitrary physical state $\psi(x)$, so that we have

$$U = A - \langle A \rangle$$ \hspace{1cm} (5A-2)

and

$$V = B - \langle B \rangle$$ \hspace{1cm} (5A-3)

where

$$\langle A \rangle = \int_{-\infty}^{\infty} dx \psi^*(x)A\psi(x)$$

and so on. Given an arbitrary wave function $\psi(x)$, let us define

$$\phi(x) = (U + i\lambda V)\psi(x)$$ \hspace{1cm} (5A-4)

with $\lambda$ real. Whatever this function is, it will be true that

$$I(\lambda) = \int_{-\infty}^{\infty} dx \phi^*(x)\phi(x) \geq 0$$ \hspace{1cm} (5A-5)

This means, because of the hermiticity of $U$ and $V$, that

$$I(\lambda) = \int_{-\infty}^{\infty} dx((U + i\lambda V)\psi(x))^*(U + i\lambda V)\psi(x)$$

$$= \int_{-\infty}^{\infty} dx \psi^*(x)(U^+ - i\lambda V^+)(U + i\lambda V)\psi(x)$$

$$= \int_{-\infty}^{\infty} dx \psi^*(x)(U - i\lambda V)(U + i\lambda V)\psi(x)$$
It follows that
\[ I(\lambda) = \int_{-\infty}^{\infty} \psi^*(x)(U^2 + i\lambda[U, V] + \lambda^2 V^2)\psi(x) \]
(5A-6)
This will have its minimum value when
\[ \frac{dI(\lambda)}{d\lambda} = 0 \]
(5A-7)
that is, when
\[ i[U, V] + 2\lambda\langle V^2 \rangle = 0 \]
When
\[ \lambda_{\text{min}} = -\frac{i[U, V]}{2\langle V^2 \rangle} \]
(5A-8)
is substituted into equation (5A-6) in the form
\[ I(\lambda_{\text{min}}) \geq 0 \]
(5A-9)
we get
\[ \langle U^2 \rangle \langle V^2 \rangle \geq \frac{1}{4} \langle i[U, V] \rangle^2 \]
(5A-10)
or equivalently,
\[ (\Delta A)^2(\Delta B)^2 \geq \frac{1}{4} \langle i[A, B] \rangle^2 \]
(5A-11)
For the operators \( p \) and \( x \) for which
\[ [p, x] = -i\hbar \]
(5A-12)
this leads to
\[ \Delta p \Delta x \geq \frac{\hbar}{2} \]
(5A-13)
Note that if for \( \psi(x) \) we take an eigenstate of the operator \( A \), for example, then
\[ (\Delta A)^2 = \int_{-\infty}^{\infty} u_n^*(x)A^2u_n(x) - \left( \int_{-\infty}^{\infty} u_n^*(x)Au_n(x) \right)^2 \]
\[ = a^2 \int_{-\infty}^{\infty} u_n^*(x)u_n(x) - \left( a \int_{-\infty}^{\infty} u_n^*(x)u_n(x) \right)^2 = 0 \]
There is no problem, since the right side of the equation also vanishes:
\[ \langle [A, B] \rangle = \int_{-\infty}^{\infty} u_n^*(x)(AB - BA)u_n(x) = \int_{-\infty}^{\infty} dx(Au_n(x))^*B - BAu_n(x) \]
\[ = a \int_{-\infty}^{\infty} u_n^*(x)Bu_n(x) - \int_{-\infty}^{\infty} dx u_n^*(x)Bau_n(x) = 0 \]
We stress again that in this derivation no use was made of wave properties, \( x \)-space or \( p \)-space wave functions, or particle-wave duality. Our result depends entirely on the operator properties of the observables \( A \) and \( B \).