Supplement 7-A

Rotational Invariance

In this supplement we show that the assumption of a central potential implies the conservation of angular momentum. We make use of invariance under rotations. The kinetic energy, which involves $\mathbf{p}^2$, is independent of the direction in which $\mathbf{p}$ points. The central potential $V(r)$ is also invariant under rotations. We show that this invariance implies the conservation of angular momentum.

INVARIANCE UNDER ROTATIONS ABOUT THE *z*-AXIS

Consider the special case of a rotation through an angle θ about the z-axis: With

$$\begin{aligned} x' &= x \cos\theta - y \sin\theta \\ y' &= x \sin\theta + y \cos\theta \end{aligned} \tag{7A-1}$$

it is easy to see that

$$r' = (x'^2 + y'^2 + z'^2)^{1/2} = (x^2 + y^2 + z^2)^{1/2} = r \tag{7A-2}$$

and

$$\begin{aligned} \left(\frac{\partial}{\partial x'}\right)^2 + \left(\frac{\partial}{\partial y'}\right)^2 &= \left(\cos\theta \frac{\partial}{\partial x} - \sin\theta \frac{\partial}{\partial y}\right)^2 + \left(\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y}\right)^2 \\ &= \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 \end{aligned} \tag{7A-3}$$

Since the Hamiltonian has an invariance property, we expect a conservation law, as we saw in the case of parity. To identify the operators that commute with H, let us consider an infinitesimal rotation about the z-axis. Keeping terms of order θ only so that

$$\begin{aligned} x' &= x - \theta y \\ y' &= y + \theta x \end{aligned} \tag{7A-4}$$

we require that

$$Hu_E(x - \theta y, y + \theta x, z) = Eu_E(x - \theta y, y + \theta x, z) \tag{7A-5}$$

If we expand this to first order in θ and subtract from it

$$Hu_E(x, y, z) = Eu_E(x, y, z) \tag{7A-6}$$

we obtain from the term linear in θ

$$H\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)u_E(x, y, z) = E\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)u_E(x, y, z) \tag{7A-7}$$

The right side of this equation may be written as

$$\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)E\, u_E(x, y, z) = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)H\, u_E(x, y, z) \qquad \text{(7A-8)}$$

If we define

$$L_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = xp_y - yp_x \qquad \text{(7A-9)}$$

then (7A-7) and (7A-8) together read

$$(HL_z - L_zH)u_E(x, y, z) = 0$$

Since the $u_E(\mathbf{r})$ form a complete set, this implies the operator relation

$$[H, L_z] = 0 \qquad \text{(7A-10)}$$

holds. L_z is the z-component of the operator

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad \text{(7A-11)}$$

which is the angular momentum. Had we taken rotations about the x- and y-axis, we would have found, in addition, that

$$[H, L_x] = 0$$
$$[H, L_y] = 0 \qquad \text{(7A-12)}$$

Thus the three components of the angular momentum operators commute with the Hamiltonian; that is, the angular momentum is a constant of the motion. This parallels the classical result that central forces imply conservation of the angular momentum.

Supplement 7-B

Angular Momentum in Spherical Coordinates

We start with spherical coordinates, as defined in Fig. 7-2. We have

$$\begin{aligned} x &= r \sin\theta \cos\varphi \\ y &= r \sin\theta \sin\varphi \\ z &= r \cos\theta \end{aligned} \tag{7B-1}$$

From this it follows that

$$\begin{aligned} dx &= \sin\theta \cos\varphi \, dr + r \cos\theta \cos\varphi \, d\theta - r \sin\theta \sin\varphi \, d\varphi \\ dy &= \sin\theta \sin\varphi \, dr + r \cos\theta \sin\varphi \, d\theta + r \sin\theta \cos\varphi \, d\varphi \\ dz &= \cos\theta \, dr - r \sin\theta \, d\theta \end{aligned} \tag{7B-2}$$

These may be solved to give

$$\begin{aligned} dr &= \sin\theta \cos\varphi \, dx + \sin\theta \sin\varphi \, dy + \cos\theta \, dz \\ d\theta &= \frac{1}{r} (\cos\theta \cos\varphi \, dx + \cos\theta \sin\varphi \, dy - \sin\theta \, dz) \\ d\varphi &= \frac{1}{r \sin\theta} (-\sin\varphi \, dx + \cos\varphi \, dy) \end{aligned} \tag{7B-3}$$

With the help of these we obtain

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial x} \frac{\partial}{\partial \varphi} \\ &= \sin\theta \cos\varphi \frac{\partial}{\partial r} + \frac{1}{r} \cos\theta \cos\varphi \frac{\partial}{\partial \theta} - \frac{\sin\varphi}{r \sin\theta} \frac{\partial}{\partial \varphi} \\ \frac{\partial}{\partial y} &= \sin\theta \sin\varphi \frac{\partial}{\partial r} + \frac{1}{r} \cos\theta \sin\varphi \frac{\partial}{\partial \theta} + \frac{\cos\varphi}{r \sin\theta} \frac{\partial}{\partial \varphi} \\ \frac{\partial}{\partial z} &= \cos\theta \frac{\partial}{\partial r} - \frac{1}{r} \sin\theta \frac{\partial}{\partial \theta} \end{aligned} \tag{7B-4}$$

We may use these to calculate the angular momentum operators in terms of the spherical angles. We have

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}_{\text{op}} = \frac{\hbar}{i} (\mathbf{r} \times \nabla) \tag{7B-5}$$

$$L_z = \frac{\hbar}{i} \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$

$$= \frac{\hbar}{i}\left[r\sin\theta\cos\varphi\left(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\right.$$

$$\left. - r\sin\theta\sin\varphi\left(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\right]$$

$$= \frac{\hbar}{i}\frac{\partial}{\partial\varphi} \tag{7B-6}$$

Similarly, we construct

$$\begin{aligned} L_x &= \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \\ &= \frac{\hbar}{i}\left[r\sin\theta\sin\varphi\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\right. \\ &\qquad \left. - r\cos\theta\left(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\right] \\ &= \frac{\hbar}{i}\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cot\,\theta\cos\varphi\frac{\partial}{\partial\varphi}\right) \end{aligned} \tag{7B-7}$$

and

$$\begin{aligned} L_y &= \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \\ &= \frac{\hbar}{i}\left[r\cos\theta\left(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\right. \\ &\qquad \left. - r\sin\theta\cos\varphi\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\right] \\ &= \frac{\hbar}{i}\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\,\theta\sin\varphi\frac{\partial}{\partial\varphi}\right) \end{aligned} \tag{7B-8}$$

It is fairly straightforward to calculate

$$\begin{aligned} \mathbf{L}^2 &= L_x^2 + L_y^2 + L_z^2 \\ &- \hbar^2\left[\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cot\,\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cot\,\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\right. \\ &\left. + \left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\,\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\,\theta\sin\varphi\frac{\partial}{\partial\varphi}\right) + \frac{\partial^2}{\partial\varphi^2}\right] \end{aligned}$$

We leave it to the reader to do the algebra. The final result is

$$\mathbf{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\,\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) \tag{7B-9}$$

Which is just the expression in eq. (7-13).

The equation

$$\mathbf{L}^2 Y(\theta, \varphi) = \hbar^2\lambda Y(\theta, \varphi)$$

when written out in spherical coordinates, is

$$\left(\frac{\partial^2}{\partial\theta^2} + \cot\,\theta\,\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \lambda\right)Y(\theta, \varphi) = 0 \tag{7B-10}$$

Let us separate variables again. If we write the solution in the form

$$Y(\theta, \varphi) = P(\theta)\Phi(\varphi) \tag{7B-11}$$

We multiply everything by $\sin^2\theta$, and divide by $P(\theta)\Phi(\varphi)$. This leads to

$$\frac{1}{P(\theta)}\left(\sin^2\theta\,\frac{d^2}{d\theta^2} + \sin\theta\cos\theta\,\frac{d}{d\theta} + \lambda\,\sin^2\theta\right)P(\theta) = -\frac{1}{\Phi(\varphi)}\frac{d^2\Phi(\varphi)}{d\varphi^2}$$

The two sides of the equation depend on different variables. They must therefore each be constant. We call the constant m^2, without specifying whether this quantity is real or complex. The solution of

$$\frac{\partial^2}{\partial\varphi^2}\Phi(\varphi) = -m^2\Phi(\varphi) \tag{7B-12}$$

is

$$\Phi(\varphi) = Ce^{\pm im\varphi} \tag{7B-13}$$

The requirement that the solution is single valued—that is, it does not change when $\varphi \to \varphi + 2\pi$—requires that m be an integer. When this is substituted into our differential equation, we end up with

$$\left(\frac{d^2}{d\theta^2} + \cot\theta\,\frac{d}{d\theta} + \lambda\right)P(\theta) = \frac{m^2}{\sin^2\theta}P(\theta) \tag{7B-14}$$

We now define

$$z = \cos\theta \tag{7B-15}$$

Using

$$\frac{d}{d\theta} = \frac{dz}{d\theta}\frac{d}{dz} = -\sin\theta\,\frac{d}{dz}$$

$$\frac{d^2}{d\theta^2} = \frac{d}{d\theta}\left(-\sin\theta\,\frac{d}{dz}\right) = -\cos\theta\,\frac{d}{dz} + \sin^2\theta\,\frac{d^2}{dz^2}$$

and using the fact that $\sin^2\theta = 1 - z^2$, can rewrite the equation as follows:

$$\left[(1 - z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} - \frac{m^2}{1 - z^2} + \lambda\right]P(z) = 0 \tag{7B-16}$$

Consider first the $m = 0$ case. Let us write the solution as

$$P(z) = \sum_{n=0}^{\infty} a_n z^n \tag{7B-17}$$

Some simple manipulations show that the coefficients a_n obey the recurrence relation

$$a_{n+2} = \frac{n^2 + n - \lambda}{(n + 1)(n + 2)}a_n \tag{7B-18}$$

The series will not terminate if λ is not an integer. In that case, for large n,

$$\frac{a_{n+2}}{a_n} \to 1 \tag{7B-19}$$

This means that for some large value of $n = N$ the series approaches a polynomial in z plus

$$1 + z + z^2 + z^3 + \cdots = \frac{1}{1 - z} \tag{7B-20}$$

This, however, is singular at $z = 1$. The only way to evade this singularity is to choose the numerator in (7B-18) to become zero when n reaches some integral value—say, $n = l$. This means that the eigenvalue is

$$\lambda = l(l + 1) \tag{7B-21}$$

and that $P(z)$ is a *polynomial of order l* in the variable z. We will label the polynomial as $P_l(z)$. These polynomials are known as *Legendre* polynomials. A short list follows:

$$\begin{aligned} P_0(z) &= 1 \\ P_1(z) &= z \\ P_2(z) &= \tfrac{1}{2}(3z^2 - 1) \\ P_3(z) &= \tfrac{1}{2}(5z^3 - 3z) \\ P_4(z) &= \tfrac{1}{8}(35z^4 - 30z^2 + 3) \end{aligned} \tag{7B-22}$$

We next observe that the $m \neq 0$ solutions are related to the $m = 0$ solutions. Let us first write the solution of (7B-16) as $P_l^m(z)$. The equation for the $P_l^m(z)$ may be written in the form

$$\frac{d}{dz}\left[(1 - z^2)\frac{dP_l^m(z)}{dz}\right] + \left[l(l + 1) - \frac{m^2}{1 - z^2}\right]P_l^m(z) = 0 \tag{7B-23}$$

If one writes

$$P_l^m(z) = (1 - z^2)^{m/2}F(z) \tag{7B-24}$$

then

$$\frac{dP_l^m(z)}{dz} = -mz(1 - z^2)^{m/2-1}F + (1 - z^2)\frac{dF}{dz}$$

and then

$$\begin{aligned} \frac{d}{dz}\left[(1 - z^2)\frac{dP_l^m(z)}{dz}\right] &= (1 - z^2)^{m/2+1}\frac{d^2F}{dz^2} - 2z(m + 1)(1 - z^2)^{m/2}\frac{dF}{dz} \\ &\quad + [m(m + 1)z^2 - 1](1 - z^2)^{m/2-1}F \end{aligned}$$

This is to be set equal to

$$\left[\frac{m^2}{1 - z^2} - l(l + 1)\right](1 - z^2)^{m/2}F$$

After this is done, some rearrangements are made, and the resulting equation is multiplied by $(1 - z^2)^{-m/2}$, one finally obtains the equation

$$(1 - z^2)\frac{d^2F}{dz^2} - 2z(m + 1)\frac{dF}{dz} + (l - m)(l + m + 1)F = 0 \tag{7B-25}$$

One can now show that this equation is satisfied by

$$F(z) = \left(\frac{d}{dz}\right)^m P_l(z) \tag{7B-26}$$

We have therefore shown that

$$P_l^m(z) = (1 - z^2)^{m/2} \left(\frac{d}{dz}\right)^m P_l(z) \tag{7B-27}$$

These functions are known as *associated Legendre functions*.

Note that only m^2 appears in the equation, so that for m in the last equation we can equally well write $|m|$. It is also clear from the form that $|m| \leq l$, since the highest power in $P_l(z)$ is z^l. For m negative we take $(-1)^m P_l^{|m|}(z)$.